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## Families of weighing matrices

Anthony V. Geramita

Norman J. Pullman

Jennifer Seberry

*University of Wollongong*, [jennie@uow.edu.au](mailto:jennie@uow.edu.au)

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## Families of weighing matrices

### Abstract

A weighing matrix is an  $n \times n$  matrix  $W = W(n, k)$  with entries from  $\{0, 1, -1\}$ , satisfying  $WW^t = kI_n$ . We shall call  $k$  the degree of  $W$ . It has been conjectured that if  $n \equiv 0 \pmod{4}$  then there exist  $n \times n$  weighing matrices of every degree  $k < n$ .

We prove the conjecture when  $n$  is a power of 2. If  $n$  is not a power of two we find an integer  $t < n$  for which there are weighing matrices of every degree

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## Families of weighing matrices

**Anthony V. Geramita, Norman J. Pullman,  
and Jennifer S. Wallis**

A weighing matrix is an  $n \times n$  matrix  $W = W(n, k)$  with entries from  $\{0, 1, -1\}$ , satisfying  $WW^t = kI_n$ . We shall call  $k$  the *degree* of  $W$ . It has been conjectured that if  $n \equiv 0 \pmod{4}$  then there exist  $n \times n$  weighing matrices of every degree  $k \leq n$ .

We prove the conjecture when  $n$  is a power of 2. If  $n$  is not a power of two we find an integer  $t < n$  for which there are weighing matrices of every degree  $\leq t$ .

Taussky [1] suggested the following generalization of Hadamard matrices:

A weighing matrix is an  $n \times n$  matrix  $W = W(n, k)$  with entries from  $\{0, 1, -1\}$ , satisfying  $WW^t = kI_n$ . We shall call  $k$  the *degree* of  $W$ . In [3, p. 433], it was conjectured that

(\*) If  $n \equiv 0 \pmod{4}$  then there exist  $n \times n$  weighing matrices of every degree  $k \leq n$ .

(Note that an  $n \times n$  weighing matrix of degree  $n$  is an Hadamard matrix and so (\*) is a generalization of the conjecture on the existence of Hadamard matrices of order  $n$  for every  $n \equiv 0 \pmod{4}$ .)

In [2] the validity of (\*) was established for  $n \in \{4, 8, 12, 16, 20, 24, 28, 32, 40\}$  and partial results were obtained

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for  $n \in \{36, 44, 52, 56\}$  in that sets of values of  $k$  were obtained for which  $W(n, k)$  exists.

For all  $n$  let  $g(n)$  be the maximum degree  $q$  for which there exist weighing matrices  $W(n, k)$  for all degrees  $k \leq q$ . Thus, conjecture (\*) is equivalent to:

$$(*) \quad g(n) = n \text{ for all } n \equiv 0 \pmod{4}.$$

The methods of [2] can be used to show that  $g(2^n) \geq 34$  for all  $n > 5$ . We show [Corollary 2 to our theorem] that in fact  $g(2^n) = 2^n$  for all  $n$  and hence establish (\*) for all powers of 2. As another corollary to the theorem we show that  $g(2^k n) \geq 2^k$  for all odd  $n$  and all  $k \geq 1$ . This is better, asymptotically, than results obtained by the methods of [2].

Call  $\{M_1, M_2, \dots, M_m\}$  an  $M$ -family of order  $n$  if for each  $i$ ,  $1 \leq i \leq m$ :

(1)  $M_i$  is a weighing matrix of order  $n$  and degree  $i$ , and

$$(2) \quad M_i M_m^t = M_m M_i^t.$$

Let  $\mu(n)$  be the largest  $m$  for which an  $M$ -family of order  $n$  exists. Evidently  $g(n) \geq \mu(n)$ .

**THEOREM.** If  $\mu(n) \geq m$  then  $\mu(2n) \geq 2m$ .

**Proof.** Suppose  $\{M_1, M_2, \dots, M_m\}$  is an  $M$ -family of order  $n$ ,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad I_p \text{ is the } p \times p \text{ identity matrix.}$$

Define

$$(a) \quad \bar{M}_i = I_2 \otimes M_i \text{ for each } i, \quad 1 \leq i \leq m,$$

$$(b) \quad \bar{M}_{m+i} = \bar{M}_i + A \otimes M_m \text{ for each } i, \quad 1 \leq i \leq m-1, \text{ and}$$

$$(c) \quad \bar{M}_{2m} = H \otimes M_m.$$

It is easily verified that  $\{\bar{M}_1, \bar{M}_2, \dots, \bar{M}_{2m}\}$  is an  $M$ -family of order  $2n$ . The matrices defined in (a) and (c) satisfy (1) and (2) because the

$M_i$  do. The matrices defined in (b) satisfy (1) because the Hadamard product of  $A$  and  $I_2$  being the zero matrix implies they are

$(1, -1, 0)$ -matrices, and  $\overline{M}_{m+i} \overline{M}_{m+i}^t = (m+i)I_{2n}$  because  $A$  is skew

symmetric; they satisfy (2) because  $HA^t = AH$ .

COROLLARY 1.  $\mu(2^k) = 2^k$  for all integers  $k \geq 1$ .

Proof.  $\{I_2, H\}$  is an  $M$ -family of order 2.

COROLLARY 2.  $g(2^k) = 2^k$  for all integers  $k \geq 1$ .

COROLLARY 3. (\*) is true for all powers of 2.

COROLLARY 4.  $g(2^k n) \geq 2^k$  for all integers  $n$  and  $k \geq 1$ .

Proof. Each matrix  $I_n \otimes M_i$  is a weighing matrix of order  $nm$  and degree  $i$  if  $M_i$  is a weighing matrix of order  $n$  and degree  $i$ .

Lemma 1 (i), 2 (i) and (iii) of [2] imply immediately that

(†) If (\*) holds for  $n$  then  $g(2^t n) \geq n + 2t$  for all integers  $t \geq 0$ .

But Corollary 4 gives far better estimates of  $g(2^t n)$  than does (†) for all sufficiently large  $t$ . For example, the results of [2] and (†) give us  $g(2^t 24) \geq 24 + 2t$  but Corollary 4 gives us  $g(2^t 24) \geq 2^{t+3}$  which is a better estimate for all  $t \geq 2$ .

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Department of Mathematics,  
Queen's University,  
Kingston,  
Ontario,  
Canada.